On the approximation by continued fractions

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1. THE APPROXIMATION CONSTANTS

Let $x \in \Omega := [0,1] \setminus \mathbb{Q}$. The expansion of x as a regular continued fraction is denoted by

$$(1.1) \qquad x = [0; a_1, a_2, a_3, \dots]$$

and the corresponding sequence of convergents by

$$\left(\frac{p_n}{q_n}\right)_{n \geq -1}.$$

The operator $T: \Omega \to \Omega$ is defined by

$$Tx := \frac{1}{x} - \left[\frac{1}{x}\right].$$

Hence, if x has the expansion (1.1) then

$$Tx = [0; a_2, a_3, a_4, \dots].$$

Therefore T is called the one-sided shift operator connected with the regular continued fraction. Finally we introduce the sequence of approximation constants $(\theta_n)_{n \geq -1}$, with

$$(1.2) \qquad \theta_n := q_n |q_n x - p_n|, \; n \geq -1.$$

As is well-known, this is a sequence in the unit interval. The dependence of a_n, p_n, q_n and θ_n on x will be disregarded in our notations.

A classical result on the approximation constants is the following theorem of Borel:

(1.3) THEOREM. *For all irrational numbers x and all $n \geq 0$:*

$$\min(\theta_{n-1}, \theta_n, \theta_{n+1}) < \frac{1}{\sqrt{5}};$$

the constant $1/\sqrt{5}$ can not be replaced by a smaller one.

Recently, Sh. Ito and H. Nakada, [6] and H.G. Kopetzky and F.J. Schnitzer, [14] proved this theorem by a new method, based upon a certain geometrical representation. Earlier, W.B. Jurkat and A. Peyerimhoff, [12] gave a proof of this theorem by using a formula that expresses θ_{n+1} as a function of θ_{n-1} and θ_n. As we will see presently, this formula is closely connected with the geometrical representation. In [14], various other classical results on the θ_n are proved in this way. In fact, the method is based upon the two-sided shift operator $\bar{T}$ from the natural extension of the ergodic system formed by Ω, the Gauss measure and the one-sided shift T, see [18].

This operator $\bar{T}$ is defined as follows. Put $\bar{\Omega} := \Omega \times [0, 1]$. Then $\bar{T}: \bar{\Omega} \to \bar{\Omega}$

$$\bar{T}(x, y) := \left(Tx, \frac{1}{y+a_1}\right) = \left(\frac{1}{x} - a_1, \frac{1}{y+a_1}\right), \tag{1.4}$$

where a_1 is the first partial quotient in the development (1.1) of x.

Using the fact that

$$\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, \dots, a_1], \; n \geq 0$$

one sees that

$$\bar{T}^n(x, 0) = \left(T^n x, \frac{q_{n-1}}{q_n}\right), \; n \geq 0.$$

Put

$$V_a := \left\{(x, y); (x, y) \in \bar{\Omega}, \frac{1}{a+1} < x < \frac{1}{a}\right\}, \; a \in \mathbb{N}$$

and

$$H_a := \left\{(x, y); (x, y) \in \bar{\Omega}, \frac{1}{a+1} < y < \frac{1}{a}\right\}, \; a \in \mathbb{N}.$$

One easily checks that

$$\bar{T} V_a = H_a$$

and that

$$\bar{T}^n(x, y) \in V_a \Leftrightarrow a_{n+1} = a, \; n \geq 0,$$

$$\bar{T}^n(x, y) \in H_a \Leftrightarrow a_n = a, \; n \geq 1,$$

where the a_n and a_{n+1} are from the representation (1.1) of x.

Now consider the function $\psi : [0,1]^2 \to \mathbb{R}^2$, defined by

$$\psi(x,y) := \left(\frac{y}{1+xy}, \frac{x}{1+xy}\right) =: (\alpha, \beta).$$

This function ψ is a C^1-diffeomorphism between the interior of the unit square and the interior of the triangle Δ with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The reason to introduce it lies in the fact that one has

$$(1.5) \qquad \psi\left(T^n x, \frac{q_{n-1}}{q_n}\right) = (\theta_{n-1}, \theta_n), \ n \geq 0,$$

see [9, formulae (2.1) and (2.2)].

The inverse of ψ is given by

$$(1.6) \qquad \psi^{-1}(\alpha, \beta) = \left(\frac{1-\sqrt{(1-4\alpha\beta)}}{2\alpha}, \frac{1-\sqrt{(1-4\alpha\beta)}}{2\beta}\right) = (x, y).$$

Further we note that $V_a' := \psi V_a$ is a quadrangle with vertices

$$\left(0, \frac{1}{a}\right), \left(\frac{a}{a+1}, \frac{1}{a+1}\right), \left(\frac{a+2}{a+2}, \frac{1}{a+2}\right) \text{ and } \left(0, \frac{1}{a+1}\right)$$

and that $H_a' := \psi H_a$ is the reflection of V_a' in $\alpha = \beta$. For $a = 1$ both quadrangles are in fact triangles.

Next define the operator $F : \Delta \to \Delta$ by putting

$$F := \psi \bar{T} \psi^{-1}.$$

A simple calculation, using (1.6) shows that we have

$$(1.7) \qquad (\alpha, \beta) \in V_a' \Rightarrow F(\alpha, \beta) = (\beta, \alpha + a\sqrt{(1-4\alpha\beta)} - a^2\beta).$$

From (1.5) and the definition of F it follows that

$$F(\theta_{n-1}, \theta_n) = (\theta_n, \theta_{n+1}), \ n \geq 0$$

and thus (1.7) shows that we have

$$(1.8) \qquad \theta_{n+1} = \theta_{n-1} + a_{n+1}\sqrt{1-4\theta_{n-1}\theta_n} - a_{n+1}^2 \theta_n, \ n \geq 0,$$

This formula, which expresses θ_{n+1} as a function of its two predecessors was also found by W.B. Jurkat and A. Peyerimhoff, [12].

In [9], [10] it was shown that

$$(\Delta, \mu, F)$$

with μ the measure defined by

$$\mu(E) := \frac{1}{\log 2} \int_E \frac{d\alpha d\beta}{\sqrt{1-4\alpha\beta}}, \ E \subset \Delta, \ E \text{ measurable},$$

is an ergodic system, from which various metrical results on the approximation constants were derived. Here we will only use arithmetic properties of the operator F.

Let $(x, y) \in H_a$. Then

$$\bar{T}^{-1}(x, y) = \left(\frac{1}{x+a}, \frac{1}{y} - a\right).$$

Comparing this with (1.4) it follows at once that one has the following converse of (1.7):

$$(\alpha, \beta) \in H_a' \Rightarrow F^{-1}(\alpha, \beta) = (\beta + a\sqrt{(1-4\alpha\beta)} - a^2\alpha, \alpha).$$

Hence

$$\theta_{n-1} = \theta_{n+1} + a_{n+1}\sqrt{1-4\theta_n\theta_{n+1}} - a_{n+1}^2\theta_n, \quad n \geq 0.$$

We recapitulate (1.8) and (1.9) in the next theorem.

(1.10) THEOREM. *Let the irrational number x have the continued fraction expansion*

$$x = [0; a_1, a_2, a_3, \ldots]$$

and define the approximation constants θ_n, $n \geq -1$, by (1.2). *Then*

$$\theta_{n+1} = \theta_{n-1} + a_{n+1}\sqrt{1-4\theta_{n-1}\theta_n} - a_{n+1}^2\theta_n,$$

$$\theta_{n-1} = \theta_{n+1} + a_{n+1}\sqrt{1-4\theta_n\theta_{n+1}} - a_{n+1}^2\theta_n, \quad n \geq 0.$$

2. THE THEOREMS OF BOREL AND FUJIWARA AND THEIR GENERALIZATIONS

In this section we will give a proof of Borel's theorem (1.3), of a theorem due to Fujiwara and of their generalizations by Bagemihl and McLaughlin. The proofs will be in the vein of those of Kopetzky and Schnitzer in [14], using however (1.8) instead of (1.4).

Denote the second coordinate function of the F from (1.7) by f, i.e.

(2.1) $$(\alpha, \beta) \in V_a' \Rightarrow f(\alpha, \beta) = \alpha + a\sqrt{(1-4\alpha\beta)} - a^2\beta.$$

Then we have on each V_a'

(2.2) $$\frac{\partial}{\partial\alpha} f(\alpha, \beta) < 0, \quad \frac{\partial}{\partial\beta} f(\alpha, \beta) < 0.$$

The latter inequality is clear immediately, the former follows after some calculations.

On each V_a, the operator $\bar{T}$ has exactly one fixed point, viz. (ξ_a, ξ_a), with

$$\xi_a = [0; a, a, a, \ldots] = \tfrac{1}{2}(-a + \sqrt{a^2+4}).$$

This corresponds with the fixed point $\psi(\xi_a, \xi_a) =: (\xi_a', \xi_a')$, with

$$\xi_a' = \frac{1}{\sqrt{a^2+4}}$$

for F in V_a'.

From this, (1.8) and (2.2) one sees at once that for each irrational number x and every integer $n \geq 0$ one has

$$(2.3) \qquad \min(\theta_{n-1}, \theta_n, \theta_{n+1}) < \frac{1}{\sqrt{(a_{n+1}^2+4)}}$$

and

$$(2.4) \qquad \max(\theta_{n-1}, \theta_n, \theta_{n+1}) > \frac{1}{\sqrt{(a_{n+1}^2+4)}}.$$

Inequality (2.3) is the generalization of Borel's result, theorem (1.3), given by N. Obrechkoff, [19] and by F. Bagemihl and J.R. McLaughlin, [1], whereas (2.4) is due to Jingcheng Tong, [25]. In the same way one can prove without any effort a generalization of Vahlen's theorem.

The line $\alpha=\beta$ cuts the upper side of V'_a in (c_a, c_a), where c_a is an abbreviation of $a/(a^2+1)$. Now, if $\theta_n > c_a$, $a_{n+1} \geq a$ then (θ_{n-1}, θ_n) must be in the shaded area of V'_a, see figure 1. This implies $\theta_{n-1} < c_a$, and with (1.8) and (2.2) we find that $\theta_{n+1} < f(0, c_a) = c_a$. Thus, if x has the expansion (1.1), then

$$(2.5) \qquad \theta_n > \frac{a_{n+1}}{a_{n+1}^2+1} \Rightarrow \theta_{n-1}, \theta_{n+1} < \frac{a_{n+1}}{a_{n+1}^2+1}, \; n \geq 0.$$

Since $a_{n+1} \geq 1$ for all n, this yields Vahlen's well-known theorem

$$\min(\theta_{n-1}, \theta_n) < \tfrac{1}{2}$$

If $a_{n+1} \neq 1$ we find Fujiwara's result, [4]:

$$a_{n+1} \neq 1, \theta_n > 2/5 \Rightarrow \theta_{n-1}, \theta_{n+1} < 2/5.$$

The general form (2.5) is again due to Bagemihl and McLaughlin, [1]. The question of these authors, whether the constant $a_{n+1}/(a_{n+1}^2+1)$ can be replaced by a smaller one is now automatically answered in the negative, see also [13].

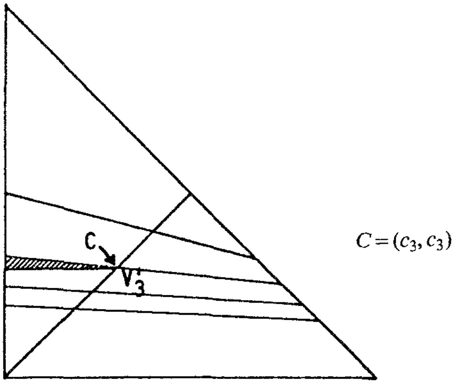

Fig. 1.

3. ON THE ARITHMETIC, GEOMETRIC AND HARMONIC MEAN OF THREE CONSECUTIVE APPROXIMATION CONSTANTS

We consider the function $G: V_a' \to \mathbb{R}$, defined by

$$G(\alpha,\beta):=\alpha\beta f(\alpha,\beta)$$

with the f from (2.1). It is not difficult to see that G takes its maximum on V_a' on the boundary. On the line segment connecting the points

$$\left(0,\frac{1}{a}\right) \text{ and } \left(\frac{a}{a+1},\frac{1}{a+1}\right)$$

one has $f(\alpha,\beta)=0$. On the segment connecting

$$\left(0,\frac{1}{a+1}\right) \text{ and } \left(\frac{a+1}{a+2},\frac{1}{a+2}\right)$$

one has

$$f(\alpha,\beta)=\frac{\alpha}{(a+1)^2}+\frac{a}{a+1},\quad 0\leq\alpha\leq\frac{a+1}{a+2},$$

whereas on the line segment connecting

$$\left(\frac{a}{a+1},\frac{1}{a+1}\right) \text{ and } \left(\frac{a+1}{a+2},\frac{1}{a+2}\right)$$

one finds that $f(\alpha,\beta)=(a+1)^2\alpha-a(a+1)$. From this it then follows easily that G takes its maximum in the point

$$\left(\frac{a+1}{a+2},\frac{1}{a+2}\right)$$

and that the maximum value is

$$\frac{(a+1)^2}{(a+2)^3}.$$

Hence we have proved the following generalization of a result of N. Macon, [17].

(3.1) THEOREM. *Let x be an irrational number with expansion* (1.1) *and let the approximation constants θ_n be defined as in* (1.2).
Then one has for every $n\geq 0$:

$$\theta_{n-1}\theta_n\theta_{n+1}<\frac{(a_{n+1}+1)^2}{(a_{n+1}+2)^3},$$

and this is best possible. In particular one has for $n\geq 0$:

$$\theta_{n-1}\theta_n\theta_{n+1}<\tfrac{4}{27}$$

where the constant 4/27 can not be replaced by a smaller one.

In exactly the same way one proves the following result:

(3.2) THEOREM. *Let x be an irrational number with expansion* (1.1) *and let the approximation constants θ_n be defined as in* (1.2).

Then one has for every $n \geq 0$:

$$\theta_{n-1}+\theta_n+\theta_{n+1}<\frac{2a_{n+1}+3}{a_{n+1}+2},$$

and this is best possible. In particular one has for $n \geq 0$:

$$\theta_{n-1}+\theta_n+\theta_{n+1}<2$$

where the constant 2 *can not be replaced by a smaller one.*

If $A(\theta_{n-1},\theta_n,\theta_{n+1})$ denotes the arithmetic mean of the three numbers θ_{n-1},θ_n and θ_{n+1}, we thus have shown that

$$A(\theta_{n-1},\theta_n,\theta_{n+1})<2/3,\ n\geq 0,$$

where 2/3 is best possible. Similarly it follows from the part of theorem (3.1) due to N. Macon that for the geometric mean G one has:

$$G(\theta_{n-1},\theta_n,\theta_{n+1})<\tfrac{1}{3}\sqrt[3]{4}\cdot n\geq 0,$$

where the constant is best possible.

For the harmonic mean of k consecutive approximation constants there is the nice result of A. Brauer and N. Macon, [3, theorem 17] which states that:

$$H(\theta_{n-1},\theta_n,\ldots,\theta_{n+k-2})<c_k$$

with

$$c_k:=\left(1+\frac{2F_k}{k}\right)^{-1},\ F_k:=\sum_{\kappa=0}^{k-1}\frac{f_\kappa}{f_{\kappa+1}},$$

where $f_0,f_1,f_2,f_3,\ldots$ is the sequencde of Fibonacci $0,1,1,2,\ldots$.

Moreover the constant c_k is best possible. For $k=3$ this gives:

$$(3.3)\qquad \frac{1}{\theta_{n-1}}+\frac{1}{\theta_n}+\frac{1}{\theta_{n+1}}>6,\ n\geq 0.$$

The following theorem gives a more precise form to (3.3):

(3.4) THEOREM. *Let x be an irrational number with expansion* (1.1) *and let the approximation constants θ_n be defined as in* (1.2).

Then one has for every $n \geq 0$:

$$\frac{1}{\theta_{n-1}}+\frac{1}{\theta_n}+\frac{1}{\theta_{n+1}}>4+a_{n+1}+\frac{2}{a_{n+1}+1}$$

and this is best possible. In particular one has for $n \geq 0$:

$$\frac{1}{\theta_{n-1}}+\frac{1}{\theta_n}+\frac{1}{\theta_{n+1}}>6$$

where the constant 6 *can not be replaced by a greater one.*

PROOF. On a line $\beta=-\alpha/c^2+1/c$, which intersects V_a' or its boundary for $a \leq c \leq a+1$, $a \in \mathbb{N}$, one has

$$f(\alpha,\beta)=a\sqrt{\left(1-\frac{2\alpha}{c}\right)^2+\alpha\left(1+\frac{a^2}{c^2}\right)}-\frac{a^2}{c}.$$

Since $1-2\alpha/c \geq 0$ on that part of this line which lies in Δ we have

$$f(\alpha,\beta)=\alpha\left(1-\frac{a}{c}\right)^2+a\left(1-\frac{a}{c}\right).$$

We introduce the function h on Δ by putting

$$h(\alpha,\beta):=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{f(\alpha,\beta)}.$$

Hence, on the line $\beta=-\alpha/c^2+1/c$ one has:

$$h(\alpha,-\alpha/c^2+1/c)=\frac{1}{\alpha}+\frac{c^2}{-\alpha+c}+\frac{1}{\alpha(1-a/c)^2+a(1-a/c)}. \tag{3.5}$$

Now

$$\frac{d}{d\alpha}\left(\frac{1}{\alpha}+\frac{c^2}{-\alpha+c}\right)=0$$

implies

$$\alpha=\frac{c}{c+1}$$

and the last term is monotonic decreasing. Therefore the function from (3.5) takes its minimum on Δ for $\alpha=c/(c+1)$, the minimum value being

$$2+\frac{1}{a+1}+\frac{1}{c}+c+\frac{1}{c-a}, \quad a \leq c \leq a+1.$$

So one sees that $h(\alpha,\beta)$ takes its minimum on V_a' in $((a+1)/(a+2), 1/(a+2))$ and that this minimum value equals

$$4+a+\frac{2}{a+1}.$$ ♦

4. THE APPROXIMATION OF A QUADRATIC IRRATIONAL NUMBER

In this section we assume that $x \in \Omega$ is a quadratic surd.

We write

$$x=[0;a_1,a_2,\ldots,a_{n_0},\overline{a_{n_0+1},\ldots,a_{n_0+h}}]. \tag{4.1}$$

Let $p(X)=aX^2+bX+c\in\mathbb{Z}[X]$ be the minimal polynomial of x, where $\operatorname{sgn}(a)$ is such, that

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}.$$

Define the binary quadratic form Q by

$$Q(X,Y)=aX^2+bXY+cY^2.$$

Finally, define the sequence of positive integers $(e_n)_{n\geq -1}$ by

(4.2) $\quad e_n:=|Q(p_n,q_n)|,\ n\geq -1,$

where the p_n and q_n are the numerators and denominators of the convergents of the continued fraction expansion (4.1) of x.

(4.3) THEOREM. *Let x be a quadratic irrational number. Denote its regular continued fraction expansion by* (4.1) *and let $(e_n)_{n\geq -1}$ be the sequence of positive integers defined in* (4.2). *Then*

(4.4) $\quad e_{n+h}=e_n,\ n\geq n_0-1;$

(4.5) $\quad e_{n+1}=e_{n-1}+a_{n+1}\sqrt{(\Delta-4e_{n-1}e_n)}-a_{n+1}^2e_n,\ n\geq n_0,$

(4.6) $\quad e_{n-1}=e_{n+1}+a_{n+1}\sqrt{(\Delta-4e_ne_{n+1})}-a_{n+1}^2e_n,\ n\geq n_0,$

with Δ the discriminant of the minimal polynomial of x;

(4.7) *for each $n\in\{n_0-1,n_0,\dots,n_0+h-2\}$ one has*

$$\lim_{m\to\infty}\theta_{n+mh}=\frac{e_n}{\sqrt{\Delta}}.$$

REMARK. Formula (4.5) is very convenient to calculate the sequence $(e_n)_{n\geq -1}$ and hence to determine the limit points of the sequence $(\theta_n)_{n\geq -1}$. Note that $\Delta-4e_{n-1}e_n$ is always a square, a fact for which we have no direct explanation.

We give a numerical example: Take

$$x:=\frac{81+\sqrt{41}}{326}=[0;3,1,2,\overline{1,2,2,1,5}];$$

then $Q(X,Y)=163X^2-81XY+10Y^2$, $\Delta=41$. One finds $e_{-1}=Q(1,0)=163$, $e_0=Q(0,1)=10$, $e_1=Q(1,3)=10$, $e_2=|Q(1,4)|=1$, $e_3=Q(3,11)=4$,

$$e_4=1+1\sqrt{41-4\cdot 1\cdot 4}-1^2\cdot 4=2,$$

$$e_5=4+2\sqrt{41-4\cdot 4\cdot 2}-2^2\cdot 2=2,$$

$$e_6=2+2\sqrt{41-4\cdot 2\cdot 2}-2^2\cdot 2=4,$$

$$e_7=2+1\sqrt{41-4\cdot 2\cdot 4}-1^2\cdot 4=1,$$

$$e_8=4+5\sqrt{41-4\cdot 4\cdot 1}-5^2\cdot 1=4.$$

Indeed $Q(p_8, q_8) = Q(211, 787) = -4$. We now see that the limit points of the sequence $(\theta_n)_{n \geq -1}$ are

$$\frac{1}{\sqrt{41}}, \frac{2}{\sqrt{41}} \text{ and } \frac{4}{\sqrt{41}}.$$

PROOF OF THEOREM (4.3). Denote the algebraic conjugate of x by x'.

In [11] a matrix

$$P = \begin{bmatrix} r & t \\ s & u \end{bmatrix}, \quad r, s, t, u \in \mathbb{Z},$$

connected with x was introduced, with the properties

(4.8) $\quad P(x) = x, P(x') = x'$, P regarded here as a Möbius transformation,

(4.9) $$P\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n+h} \\ q_{n+h} \end{pmatrix}, \quad n \geq n_0 - 1$$

and

(4.10) $\quad \det P = (-1)^h$.

Put

$$(Q \circ P)(X, Y) := Q(rX + tY, sX + uY).$$

Since

$$Q(X, Y) = a(X - xY)(X - x'Y)$$

we have

$$(Q \circ P)(X, Y) = a(rX + tY - x(sX + uY))(rX + tY - x'(sX + uY)) =$$
$$= a(r - xs)(r - x's)\left(X + \frac{t - xu}{r - xs}\, Y\right)\left(X + \frac{t - x'u}{r - x's}\, Y\right).$$

From (4.8) we infer that

$$\frac{t - xu}{r - xs} = -x$$

and similarly for x'. Therefore

$$(Q \circ P)(X, Y) = (r - xs)(r - x's)Q(X, Y).$$

From (4.8) it also follows that

$$x + x' = \frac{r - u}{s} \text{ and } xx' = -\frac{t}{s},$$

thus we find that

$(r - xs)(r - x's) = ru - ts,$

which, by (4.10), equals $(-1)^h$.

We now have proved that

(4.11) $\quad (Q \circ P)(X, Y) = (-1)^h Q(X, Y).$

Hence, by (4.9) we find

$$Q(p_{n+h}, q_{n+h}) = (-1)^h Q(p_n, q_n),\ n \geq n_0 - 1,$$

from which (4.4) follows.

Next we observe that

$$e_n = |Q(p_n, q_n)| = |a| q_n^2 \left|\frac{p_n}{q_n} - x\right| \left|\frac{p_n}{q_n} - x'\right| = |a| \theta_n \left|\frac{p_n}{q_n} - x'\right|.$$

For $n \geq n_0 - 1$ and $m \geq 0$ we thus have, using the just proved (4.4):

$$e_{n+mh} = e_n = |a| \theta_{n+mh} \left|\frac{p_{n+mh}}{q_{n+mh}} - x'\right|.$$

By letting m go to infinity this gives

$$e_n = |a|\,|x - x'| \lim_{m \to \infty} \theta_{n+mh}.$$

from which (4.7) follows, since $|a|\,|x-x'| = \sqrt{\Delta}$.

Finally, to prove (4.5) and (4.6) we note that by (1.8) we have for $n \geq n_0$ and $m \geq 0$:

$$\theta_{n+mh+1} = \theta_{n+mh-1} + a_{n+1}\sqrt{1 - 4\theta_{n+mh-1}\theta_{n+mh}} - a_{n+1}^2 \theta_{n+mh}.$$

Letting $m \to \infty$, using (4.7) and multiplying with $\sqrt{\Delta}$ we find (4.5). The converse (4.6) is proved in the same way, using (1.9). ♦

We finish this section by some remarks on the case that $x = \sqrt{N} - [\sqrt{N}]$, with N a positive integer which is not a square. As is well-known one has

$$x = [0; \overline{a_1, \ldots, a_h}].$$

with

(4.12) $\quad a_h = 2[\sqrt{N}]$

and

(4.13) $\quad a_\eta = a_{h-\eta},\ \eta = 1, 2, \ldots, h-1.$

Of course

(4.14) $\quad e_n = |P_n^2 - NQ_n^2|,\ n \geq -1$

where $(P_n/Q_n)_{n \geq -1}$ is the sequence of convergents of $\sqrt{N}$. Note that $\Delta = 4N$.

From (4.4) we see that, with an obvious notation

(4.15) $\quad (e_n)_{n \geq -1} = (\overline{e_{-1}, e_0, \ldots, e_{h-2}})$

whereas from (4.14) it follows that $e_{-1} = 1$, $e_0 = N - [\sqrt{N}]^2$.

From (4.6) we infer that

$$\begin{aligned} e_{h-2} &= e_h + a_h\sqrt{\Delta - 4e_{h-1}e_h} - a_h^2 e_{h-1} \\ &= e_h + a_h\sqrt{\Delta - 4e_{-1}e_0} - a_h^2 e_{-1} \\ &= e_h + a_h\sqrt{4N - 4(N - [\sqrt{N}]^2)} - a_h^2 \\ &= e_h + 4[\sqrt{N}]^2 - 4[\sqrt{N}]^2 = e_h. \end{aligned}$$

From $e_{h-2} = e_h$, (4.15), (4.5) and (4.6) we now see at once that

$$e_\eta = e_{h-2-\eta}, \quad \eta = 0, 1, \ldots, h-1,$$

i.e. the part $e_0, \ldots, e_{h-2}$ of the period in (4.15) is symmetric.

Note also that $e_n + e_{n+1} \leq [2\sqrt{N}]$, $(n \geq -1)$, since $\theta_n + \theta_{n+1} < 1$.

Finally, $N - e_n e_{n+1}$ is always a square.

Denote by W the density function according to which the sequence

$$(\theta_{n-1}, \theta_n)_{n \geq 0}$$

is distributed over Δ in the (α, β) plane. The above observed symmetry shows that W is symmetric in its two variables i.e.

$$W(\alpha, \beta) = W(\beta, \alpha).$$

This symmetry it shares with the density function for the sequence

$$(\theta_{n-1}, \theta_n)_{n \geq -1}$$

of almost all x, which equals

$$\frac{1}{\log 2} \frac{1}{\sqrt{1 - 4\alpha\beta}},$$

see [9], [10].

5. THE THEOREMS OF LEGENDRE AND BOREL FOR THE NEAREST INTEGER CONTINUED FRACTION

In this section we shall prove two results on the nearest integer continued fraction. For the definition of this type of continued fraction the reader may consult Hurwitz' original paper [5], or Perron's book [20]. See also [8] and [15].

We denote by

$$\left(\frac{A_n}{B_n}\right)_{n \geq -1}$$

the sequence of convergents of the nearest integer continued fraction.

The sequence of approximation constants $(\Theta_n)_{n \geq -1}$ is defined by

$$\Theta_n := B_n |B_n x - A_n|, \quad n \geq -1. \tag{5.1}$$

Finally we use the notations

$$g := \tfrac{1}{2}(\sqrt{5}-1) \text{ and } G := \tfrac{1}{2}(\sqrt{5}+1)$$

throughout the rest of this paper.

For all irrational x and all positive integers n one has

$$\Theta_n < g \quad \text{(Hurwitz)}$$

and

$$(5.2) \qquad \begin{cases} \min(\Theta_{n-1}, \Theta_n) < 2\sqrt{5}-4 = 0.4721\ldots, \text{ (Kurosu [16], Sendov [24],} \\ \text{Kraaikamp [15]).} \end{cases}$$

These constants are best possible.

The next theorem is the analogue for the nearest integer continued fraction of Legendre's classical theorem for the regular continued fraction expansion.

(5.3) THEOREM. *Let x be an irrational number and let $(p,q) \in \mathbb{Z}^2$ such that $(p,q)=1$, $q>0$ and*

$$q|qx-p| < g^2 = \tfrac{1}{2}(3-\sqrt{5}).$$

Then there exists a non-negative integer n such that

$$\frac{p}{q} = \frac{A_n}{B_n}.$$

The constant g^2 can not be replaced by a larger one.

PROOF. Suppose that there is not such an n. Since $\tfrac{1}{2}(3-\sqrt{5}) < \tfrac{1}{2}$ there exists, by Legendre's theorem, a non negative integer n such that

$$\frac{p}{q} = \frac{p_n}{q_n},$$

where $(p_n/q_n)_{n \geq -1}$ is the sequence of regular convergents of x. From the way $(A_n/B_n)_{n \geq -1}$ forms a subsequence of $(p_n/q_n)_{n \geq -1}$ we see that

$$a_{n+1} = 1$$

and that there exists a non-negative integer k, such that

$$(5.4) \qquad \frac{A_{k-1}}{B_{k-1}} = \frac{p_{n-1}}{q_{n-1}}, \ \frac{A_k}{B_k} = \frac{p_{n+1}}{q_{n+1}}.$$

In view of $\theta_n < \tfrac{1}{2}(3-\sqrt{5}) = g^2$, the point (θ_{n-1}, θ_n) lies in V_1' under the line $\beta = g^2$. This line intersects the line $\beta = -\tfrac{1}{4}\alpha + 1/2$ in the point $(2\sqrt{5}-4, g^2)$ and therefore

$$(5.5) \qquad \theta_{n-1} > 2\sqrt{5}-4.$$

Let Γ be that part of V_1' which lies under the line $\beta=g^2$. Because of (2.2) the function f from (2.1) takes its minimum value on Γ in the point (g, g^2). Now $f(g, g^2)=2\sqrt{5}-4$, hence

$$(5.6) \qquad \theta_{n+1}>2\sqrt{5}-4.$$

But (5.5) and (5.6) are in conflict with the analogue of Vahlen's theorem for the nearest integer continued fraction, (5.2). ♦

Finally we shall give the analogue of Borel's theorem (1.3) for the nearest integer continued fraction.

(5.7) THEOREM. *For all irrational numbers x and all integers $n\geq 0$ one has:*

$$\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1})<\tfrac{5}{2}(5\sqrt{5}-11)=0.4508\ldots.$$

This constant can not be replaced by a smaller one.

Before giving the proof we make some comments. In view of (5.2) it is clear that the constant in theorem (5.7) can not be greater than $2\sqrt{5}-4$, neither can it be smaller than $1/\sqrt{5}=0.4472\ldots$, for that would be in conflict with a well-known result of Hurwitz. It is curious that the constant is slightly greater than $1/\sqrt{5}$, in fact, the difference is

$$\frac{1}{\sqrt{5}}\left(\tfrac{1}{2}(125-55\sqrt{5})-1\right).$$

The number $\frac{1}{2}(125-55\sqrt{5})=1.0081\ldots$ also comes up in the probability that Θ_{n-1} is smaller than Θ_n, see [15]. We have no explanation for this.

PROOF OF THEOREM (5.7). Put $\Omega:=(-1/2, 1/2)\setminus\mathbb{Q}$. Denote the nearest integer continued fraction expansion of $x\in\Omega$ by

$$x=\frac{\varepsilon_1|}{|b_1}+\frac{\varepsilon_2|}{|b_2}+\frac{\varepsilon_3|}{|b_3}+\ldots,\ \varepsilon_n=\pm 1, b_n\geq 2, b_n+\varepsilon_{n+1}\geq 2 \text{ for } n\geq 1.$$

Define the operator $S:\Omega\to\Omega$ by

$$Sx:=\frac{\varepsilon_1}{x}-b_1.$$

It was proved by G.J. Rieger, [21] and A.M. Rockett, [22] that the system

$$(\Omega, \mu, S)$$

with μ the measure with density function ϱ, where

$$\varrho(t):=\begin{cases}\dfrac{1}{\log G}\dfrac{1}{G+t}, & 0\leq t\leq 1/2\\[2ex] \dfrac{1}{\log G}\dfrac{1}{G+1+t}, & -1/2\leq t<0\end{cases},$$

forms an ergodic system. H. Nakada gave the natural extension of this system [18, theorem 2 with $\alpha = 1/2$]. Here we need only the operator $\bar{S}$ from this natural extension. It is given by

$$\bar{S}(x, y) := \left(Sx, \frac{1}{\varepsilon_1 y + b_1} \right) = \left(\frac{\varepsilon_1}{x} - b_1, \frac{1}{\varepsilon_1 y + b_1} \right),$$

with

$$(x, y) \in [-\tfrac{1}{2}, 0] \setminus \mathbb{Q} \times [0, g^2] \cup [0, \tfrac{1}{2}] \setminus \mathbb{Q} \times [0, g].$$

Further we need the relation

$$\frac{B_{n-1}}{B_n} = \frac{1}{|b_n} + \frac{\varepsilon_n}{|b_{n-1}} + \ldots + \frac{\varepsilon_2}{|b_1}, \quad n \geq 1$$

from which it follows that

$$\bar{S}^n(x, y) = \left(S^n x, \frac{B_{n-1}}{B_n} \right), \quad n \geq 1.$$

For the Θ_n from (5.1) we have

$$\Theta_{n-1} = \frac{v_n}{1 + s_n v_n}, \quad \Theta_n = \frac{\varepsilon_{n+1} s_n}{1 + s_n v_n}, \quad n \geq 0$$

where

$$s_n := S^n x \text{ and } v_n := \frac{B_{n-1}}{B_n}.$$

In exactly the same way as (1.8) was found one can prove the following theorem:

(5.8) THEOREM. *Let the irrational number x have the nearest integer continued fraction expansion*

$$x = \frac{\varepsilon_1}{|b_1} + \frac{\varepsilon_2}{|b_2} + \frac{\varepsilon_3}{|b_3} + \ldots$$

and define the approximation constants Θ_n, $n \geq -1$ *by* (5.1).

Then

$$\Theta_{n+1} = \varepsilon_{n+2}(\varepsilon_{n+1}\Theta_{n-1} + b_{n+1}\sqrt{1 - 4\varepsilon_{n+1}\Theta_{n-1}\Theta_n} - b_{n+1}^2\Theta_n), \quad n \geq 0.$$

The sequence $(\Theta_{n-1}, \Theta_n)_{n \geq 1}$ is distributed over the two regions Γ_1 and Γ_{-1} with Γ_1 the quadrangle with vertices $(0, 0)$, $(g, 0)$, $(2g^3, g^2)$ and $(0, 1/2)$ and Γ_{-1} the quadrangle with vertices $(0, 0)$, $(g^2, 0)$, $(2g^3, g)$ and $(0, 1/2)$. One has

$$(\Theta_{n-1}, \Theta_n) \in \Gamma_\varepsilon \text{ if and only if } \varepsilon_{n+1} = \varepsilon, \ \varepsilon \in \{-1, 1\}, \text{ see [15]}.$$

Both Γ_1 and Γ_{-1} are divided into regions with constant b_{n+1}. For Γ_1 this division is given by the lines

$$\beta = -c^2\alpha + c, \ c = \tfrac{2}{5}, \tfrac{2}{7}, \tfrac{2}{9}, \ldots,$$

whereas Γ_{-1} is divided into regions with constant b_{n+1} by the lines

$$\beta = c^2\alpha - c, \quad c = -\tfrac{2}{5}, \quad -\tfrac{2}{7}, \quad -\tfrac{2}{9}, \ldots,$$

see figure 2. We introduce for these regions the obvious notation

$$\Gamma_{\varepsilon, b}, \ \varepsilon \in \{-1, 1\}, \ b \geq 2.$$

Every $\Gamma_{\varepsilon, b}$ with $b \neq 2$ is divided into two parts, according to $\varepsilon_{n+2} = +1$ or $\varepsilon_{n+2} = -1$ by the lines

$$\beta = -c^2\alpha + c, \ c = \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \ldots, \text{ for } \varepsilon_{n+1} = +1$$

and

$$\beta = c^2\alpha - c, \ c = -\tfrac{1}{3}, -\tfrac{1}{5}, -\tfrac{1}{7}, \ldots, \text{ for } \varepsilon_{n+1} = -1.$$

These lines are dotted in figure 2.

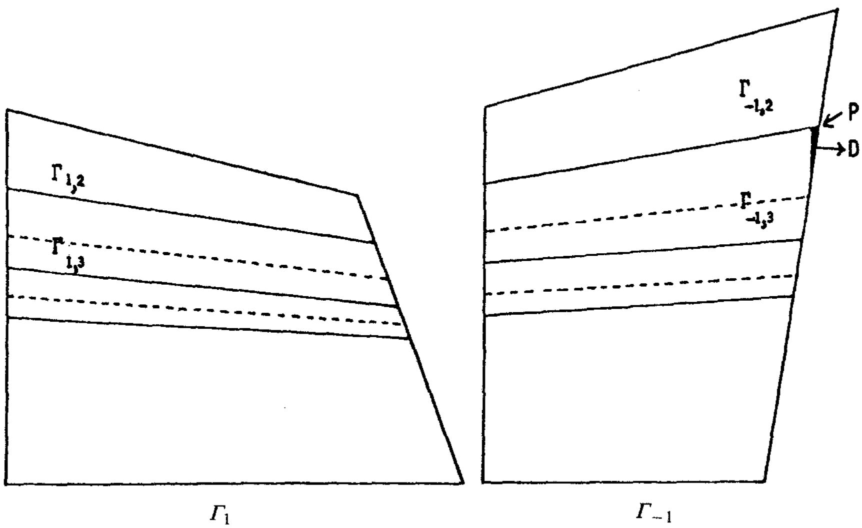

Fig. 2.

For a point $(\Theta_{n-1}, \Theta_n) \in \Gamma_1$ one always has min $(\Theta_{n-1}, \Theta_n) < 2/5$, $(2/5, 2/5)$ being the point of intersection of the lines $\beta = -\frac{1}{4}\alpha + 1/2$ and $\beta = \alpha$. So our only concern is with points in Γ_{-1}. First we remark that the point $(1/\sqrt{5}, 1/\sqrt{5})$ lies on the boundary of Γ_{-1}, more precisely on the boundary of $\Gamma_{-1,3}$. On $\Gamma_{-1,2}$ the next Θ is calculated according to the formula

$$k(\alpha, \beta) = -\alpha + 2\sqrt{1 + 4\alpha\beta} - 4\beta,$$

see theorem (5.8) with $\varepsilon_{n+2}=1$, $\varepsilon_{n+1}=-1$, $b_{n+1}=2$.

On that part of $\Gamma_{-1,2}$ where $\alpha \geq 1/\sqrt{5}$ (then β is automatically greater than $1/\sqrt{5}$) this function k takes its maximum on the boundary. The value on the line $\beta = c^2\alpha - c$ of $k(\alpha, \beta)$ equals $-\alpha(1+2c)^2 + 2(1+2c)$. From this we see that the maximum value is assumed for $\alpha = 1/\sqrt{5}$, $c = -2/5$ and that it is

$$\tfrac{2}{5} - \frac{1}{25\sqrt{5}} = 0.3821\ldots < \frac{1}{\sqrt{5}}.$$

Thus for points $(\Theta_{n-1}, \Theta_n) \in \Gamma_{-1,2}$ we always have

$$\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1}) < \frac{1}{\sqrt{5}}.$$

Hence it remains to study what happens when (Θ_{n-1}, Θ_n) lies in that part of $\Gamma_{-1,3}$ for which $\alpha \geq 1/\sqrt{5}$ and $\beta \geq 1/\sqrt{5}$, see figure 2. Call this triangle D. On D, the next Θ is calculated according to the formula

$$h(\alpha, \beta) = \alpha - 3\sqrt{1 + 4\alpha\beta} + 9\beta,$$

theorems (5.8) with $\varepsilon_{n+2} = -1$, $\varepsilon_{n+1} = -1$, $b_{n+1} = 3$.

After some calculations one finds that h attains its maximum on D in the upper right vertex P, the point with coordinates

$$P(\tfrac{5}{2}(5\sqrt{5} - 11), 2\sqrt{5} - 4),$$

and that this maximum value equals g. Hence one can not attain better than

$$\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1}) < \tfrac{5}{2}(5\sqrt{5} - 11)$$

on D. ♦

F. Schweiger, [23], has shown that in case of the continued fraction expansion with odd partial quotients there always are infinitely many approximation constants smaller than $1/\sqrt{5}$. We now show that the same holds for the nearest integer continued fraction:

(5.9) THEOREM. *Let x be an irrational number and let the sequence $(\Theta_n)_{n \geq -1}$ of nearest integer approximation constants be defined by* (5.1).
Then there are infinitely many positive integers n for which

$$\Theta_n < \frac{1}{\sqrt{5}}.$$

PROOF. Suppose there exists an n_0 such that $\Theta_{n-1} > 1/\sqrt{5}$ for $n \geq n_0$. Then $(\Theta_{n-1}, \Theta_n) \in D$ for $n \geq n_0$ and this implies that $\varepsilon_{n+1} = -1$, $b_n = 3$ for $n \geq n_0$. Hence, for $n \geq n_0$

$$S^n x = \frac{-1|}{|3} + \frac{-1|}{|3} + \frac{-1|}{|3} + \ldots = -g^2,$$

$$\frac{B_{n-1}}{B_n} < g^2.$$

This in its turn implies that (Θ_{n-1}, Θ_n) lies for $n \geq n_0$ in $\Gamma_{-1,3}$ on the line $\beta = \alpha g^4 + g^2$, but certainly not in the point $(1/\sqrt{5}, 1/\sqrt{5})$. Therefore $(\Theta_{n-1}, \Theta_n) \notin D$, which is the desired contradiction. ♦

Using the fact that the point $(-g^2, g^2)$ is a fixed point of $\bar{S}$ one can show that for each $n_0 \in \mathbb{N}$ there exists an irrational number x for which n_0 consecutive points $(\Theta_{n+k}, \Theta_{n+k+1})$, $k = 0, \ldots, n_0 - 1$, lie in D. Hence there does not exist a natural number n_0 such that

$$\min\,(\Theta_{n+1}, \ldots, \Theta_{n+n_0}) < \frac{1}{\sqrt{5}}$$

for all irrational numbers x and all $n \in \mathbb{N}$.

Dominique Barbolosi, [2], showed recently that the sequence $(A_n/B_n)_{n \geq -1}$ of nearest integer convergents always forms a subsequence of the sequence of convergents of the continued fraction with odd partial quotients. Hence, theorem (5.9) contains Schweiger's result.

REFERENCES

1. Bagemihl, F. and J.R. McLaughlin – Generalization of some classical theorems concerning triples of consecutive convergents to simple continued fractions. J. reine angew. Math. **221**, 146–149 (1966).
2. Barbolosi, Dominique – Fractions continues à quotients partiels impairs. Propriétés arithmétiques et ergodiques. Thèse, janvier 1988, Université de Provence.
3. Brauer, Alfred and Nathaniel Macon – On the approximation of irrational numbers by the convergents of their continued fractions. Amer. J. Math. **71**, 349–361 (1949) and **72**, 419–424 (1950).
4. Fujiwara, M. – Bemerkungen zur Theorie der Approximation der irrationalen Zahlen durch rationale Zahlen, Tôhoku Math. J. **14**, 109–115 (1918).
5. Hurwitz, A. – Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche. Math. Ann. **39**, 279–284 (1891).
6. Ito, Sh. and H. Nakada – On natural extensions of transformations related to Diophantine approximations. Number Theory and Combinatorics, 185–207, ed. J. Akiyama et al. World Scientific Publishing Co., Singapore (1985).
7. Ito, Shunji, Hitoshi Nakada and Shigeru Tanaka – On the invariant measure for the transformations associated with some real continued fractions. Keio Engeneering Reports **30**, 159–175 (1977).
8. Jager, H. – Metrical results for the nearest integer continued fraction. Indag. Math. **47**, 417–427 (1985).
9. Jager, H. – The distribution of certain sequences connected with the continued fraction. Indag. Math. **48**, 61–69 (1986).
10. Jager, H. – Continued fractions and ergodic theory. "Transcendental numbers and related topics". RIMS Kokyuroku **599**, Kyoto University, Kyoto, Japan, 55–59 (1986).
11. Jager, Hendrik et Pierre Liardet – Distributions arithmétiques des dénominateurs de convergents de fractions continues. Indag. Math. **50**, 181–197 (1988).
12. Jurkat, W.B. and A. Peyerimhoff – Characteristic approximation properties of quadratic irrationals. Internat. J. Math. & Math. Sci. **1**, 479–496 (1977).
13. Kopetzky, H.G. and F.J. Schnitzer – Bemerkungen zu einem Approximationssatz für regelmäßige Kettenbrüche. J. reine angew. Math. **293/294**, 437–440 (1977).

14. Kopetzky, H.G. and F.J. Schnitzer – A geometrical approach to approximations by continued fractions. J. Austr. Math. Soc. (Series A) **43**, 176–186 (1987).

15. Kraaikamp, Cor – The distribution of some sequences connected with the nearest integer continued fraction. Indag. Math. **49**, 177–191 (1987).

16. Kurosu, K. – Notes on Some Points in the Theory of Continued Fractions. Japanese J. of Math. **1**, 17–21 (1924).

17. Macon, Nathaniel – Some theorems on the approximation of irrational numbers by the convergents of their continued fractions. J. Elisha Mitchell Sci. Soc. **67**, 99–107 (1951).

18. Nakada, Hitoshi – Metrical theory for a class of continued fraction transformations and their natural extensions. Tokyo J. Math. **4**, 399–426 (1981).

19. Obrechkoff, N. – Sur l'approximation des nombres irrationnels par des nombres rationnels. C.R. Acad. Bulgare Sci. **3**, No. 1 1–4 (1951).

20. Perron, O. – Die Lehre von den Kettenbrüchen. Band I, B.G. Teubner, Stuttgart, 3. verb. u. etw. Aufl.

21. Rieger, G.J. – Mischung und Ergodizität bei Kettenbrüchen nach nächsten Ganzen. J. reine angew. Math. **310**, 171–181 (1979).

22. Rockett, A.M. – The metrical theory of continued fractions to the nearer integer. Acta Arith. **38**, 97–103 (1980).

23. Schweiger, F. – On the approximation by continued fractions with odd and even partial quotients. Arbeitsbericht Mathematisches Institut Salzburg **1–2**, 105–114 (1984).

24. Sendov, B. – Der Vahlensche Satz über die singulären Kettenbrüche und die Kettenbrüche nach nächsten Ganzen. Annuaire Univ. Sofia Fac. Sci. Phys. Math. Livre 1 Math. **54**, 251–258 (1959/60).

25. Tong, Jingcheng – The Conjugate Property of the Borel Theorem on Diophantine Approximation. Math. Z. **184**, 151–153 (1983).